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PURITY IN LOCALLY-PRESENTABLE MONOIDAL CATEGORIES

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Let X be a locally-presentable monoidal category. We show that every "small" subobject in X is contained in a small pure subobject. In the spirit of [2], this is used to analyse the structure of coalgebraic categories over X. These results complement the work of Gabriel and Ulmer and open the door to the much neglected study of coalgebras in abstract multilinear algebra.

By a monoidal category we mean a category X equipped with a bifunctor, "tensor", $\otimes : X \times X \to X$. It is assumed that $\otimes$ is coherently symmetric, associative, and unitary, with unit object I. If X is an object in X, X^n denotes the n-fold tensor of X with itself, and $X^0 = I$.

A prop is a monoidal category P having the natural numbers, N, as its object set. It is assumed that the tensor acts by addition on objects, and that the symmetric group S_n is a subset of $P(n, n)$ for all $n \in$ N. For details see Mac Lane [4]. P is said to be coalgebraic if it is generated as a monoidal category by a set of maps with domain 1.

If P is a prop and X a monoidal category, let $X^{(P)}$ denote the class of all tensor preserving functors from P to X which also preserve the natural isomorphisms required by the monoidal structures of P and X. Then $X^{(P)}$ is a category in a natural way and is equipped with the usual forgetful functor to X. A map $A \xrightarrow{x} B$ in $X^{(P)}$ is just an X-map such that for all maps $od \xrightarrow{\sigma} or$ in P the following diagram commutes:

$$\begin{tikzcd}
A^{\sigma d} \arrow[r, "x^{\sigma d}"] \arrow[d, "\sigma A"'] & B^{\sigma d} \arrow[d, "\sigma B"] \\
A^{\sigma r} \arrow[r] & B^{\sigma r}
\end{tikzcd}$$

A category of the form $X^{(P)}$ is said to be propable over X. It is coalgebraic if P is.

Recall that a diagram D in an arbitrary category X is α-filtered, where α is a regular cardinal, if:

(i) for every family $\{D_i\}_{i\in I}$ of objects in $\boldsymbol{D}$ such that $|I| < \alpha$, there exists an object D in $\boldsymbol{D}$ and maps $\{D_i \to D\}_{i\in I}$ in $\boldsymbol{D}$.

(ii) for every family of maps $\{D \xrightarrow{d_i} D'\}_{i\in I}$ in $\boldsymbol{D}$ such that $|I| < \alpha$, there exists a map $D' \xrightarrow{d} D''$ in $\boldsymbol{D}$ such that $d_i d = d_j d$ for all i and j in I.

Given a functor $\boldsymbol{X} \to \boldsymbol{Y}$, if there exists a regular cardinal α such that F preserves α-filtered colimits, we say that F has rank, and rank $F \leqslant \alpha$. For an object X in $\boldsymbol{X}$, we write $\mathrm{rank}_{\boldsymbol{X}} X$ (or rank X) instead of rank $\boldsymbol{X}(X,-)$. A category $\boldsymbol{X}$ is α-locally-presentable if it is cocomplete and has a generating set $\boldsymbol{G}$ such that rank $G \leqslant \alpha$ for all G in $\boldsymbol{G}$. An object X in $\boldsymbol{X}$ is said to be β-generated, where β is a regular cardinal, if there is a proper epimorphism $\oplus_{i\in I} G_i \to X$, where $G_i \in \boldsymbol{G}$ and $|I| < \beta$.

Locally presentable categories have been studied extensively by Gabriel and Ulmer [3], and by Barr [1]. They include the categories **Sets**, **Mod-*R***, **Cat**, categories of sheaves, the category of Banach spaces (norm decreasing maps), and many others. If $\boldsymbol{X}$ is α-locally-presentable, it is known that $\boldsymbol{X}$ is complete, that every element of $\boldsymbol{X}$ has rank, and that α-filtered colimits commute with α-limits, i.e. kernel pairs and products with fewer than α factors. Also, every element of $\boldsymbol{X}$ is the colimit of an α-filtered diagram in $\boldsymbol{G}$.

Henceforth, let $\boldsymbol{X}$ be an α-locally-presentable monoidal category. We assume that $\boldsymbol{X}$ has images and that rank $\otimes \leqslant \alpha$.

1. Lemma. *We may assume that* $\boldsymbol{G}$ *is closed under finite applications of* $\otimes$.

Recall that a subobject $X \xrightarrow{i} Y$ in $\boldsymbol{X}$ is pure if $X \otimes Z \xrightarrow{i\otimes 1} Y \otimes Z$ is a monomorphism for every Z in $\boldsymbol{X}$. We shall show that every "small" subobject is contained in a pure subobject that is not too much bigger. Using Theorem II.3.2 in [1] we may choose a regular cardinal γ having the following properties:

(1) $\gamma > \alpha + |\boldsymbol{G}|$.

(2) Every object in $\boldsymbol{X}$ is the direct colimit of its γ-generated subobjects.

(3) X in $\boldsymbol{X}$ is γ-generated if and only if $|\bigcup_{G\in\boldsymbol{G}} \boldsymbol{X}(G, X)| < \gamma$.

(4) Whenever X and Y are γ-generated, so are $X \times Y$ and $X \otimes Y$.

2. Lemma. *Let X be a γ-generated subobject of Y in $\boldsymbol{X}$. For each $G \in \boldsymbol{G}$ there exists a 2^γ-generated subobject, $X(G)$, of Y containing X and such that the kernel pair of $X \otimes G \xrightarrow{i\otimes 1} Y \otimes G$ is a kernel pair for $X \otimes G \xrightarrow{i\otimes 1} X(G) \otimes G$, where i is the inclusion.*

Proof. Let $\boldsymbol{U}_X$ (or just $\boldsymbol{U}$) denote the set of all subobjects of Y of the form $X \cup X'$ where X' is γ-generated. By the remarks preceeding the lemma we have $\varinjlim \boldsymbol{U} = Y$. Now let $K_G \rightrightarrows X \otimes G \xrightarrow{i\otimes 1} Y \otimes G$ be a kernel pair diagram, and let $K_U \rightrightarrows X \otimes G \xrightarrow{i\otimes 1} U \otimes G$ be a kernel pair for each $U \in \boldsymbol{U}$. Let $\boldsymbol{K}$ denote the set of all K_G's. Since $\boldsymbol{U}$ is closed under unions of less than α elements (since $\alpha < \gamma$), $\boldsymbol{U}$ is an α-directed set and so is $\boldsymbol{K}$. Since $Y \otimes G = \varinjlim \boldsymbol{U} \otimes G$ and kernel pairs are α-limits, the canonical map $K_G \to \varinjlim \boldsymbol{K}$ is an isomorphism. By the choice of γ, both $X \otimes G$ and $(X \otimes G) \times (X \otimes G)$ are γ-generated. Since subobjects are determined by sets of maps from the generating set, again by the choice of γ there are fewer than 2^γ isomorphism classes of subobjects of $(X \otimes G) \times (X \otimes G)$, and hence fewer than 2^γ distinct members of $\boldsymbol{K}$. For each distinct $K \in \boldsymbol{K}$, choose an object $U \in \boldsymbol{U}$ such that $K \rightrightarrows X \otimes G \to U \otimes G$ is a kernel pair and let $\boldsymbol{U}?$ denote the set of all chosen objects. Since $|\boldsymbol{U}?| < 2^\gamma$, $\boldsymbol{U}?$ has less than 2^γ subsets of cardinality α, so closing up $\boldsymbol{U}?$ under α-unions yields another subset of $\boldsymbol{U}$, denoted $\boldsymbol{U}!$, such that $|\boldsymbol{U}!| < 2^\alpha$ and $\boldsymbol{U}!$ is α-directed. Thus, defining $X(G) = \varinjlim \boldsymbol{U}!$ $X(G)$ is a subobject of Y and obviously contains X. Since $X(G)$ is the union of fewer than 2^γ γ-generated objects, it is 2^γ-generated. Finally, because $\boldsymbol{U}!$ is α-directed:

$$\begin{aligned}\ker(X \otimes G \to X(G) \otimes G) &= \ker(X \otimes G \to \varinjlim \boldsymbol{U}! \otimes G)\\ &= \varinjlim \ker(X \otimes G \to \boldsymbol{U}! \otimes G) = \varinjlim \boldsymbol{K} = K_G\,.\end{aligned}$$

Iterating the above construction would yield a subobject of Y that is pure "for" the generator G, and subsequent diagonalization would yield a pure subobject. We perform both operations in the proof of:

3. Theorem. *There exists a regular cardinal γ' such that each γ-generated subobject in $\boldsymbol{X}$ is contained in a γ'-generated pure subobject.*

Proof. Let X be a γ-generated subobject of Y. Let $\boldsymbol{E}$ denote the set of (ordered) families of elements of $\boldsymbol{G}$ with fewer than α terms. Define $X(E)$ for $E \in \boldsymbol{E}$ by ordered application of the process $(-)\,(G)$ for $G \in E$.

More precisely, let $\boldsymbol{E} = \bigcup_{|\lambda| < \alpha} \boldsymbol{G}^\lambda$, the union taken over all ordinals λ whose cardinality is less than α. If $E \in \boldsymbol{G}^\lambda$, let $E|\mu$ denote the function E restricted to $\mu < \lambda$, and define $X(E)$ as follows:

$$X(E) = \begin{cases} (X(E|\lambda - 1))\,(\lambda - 1)E & \text{if } \lambda \text{ is a non-limit ordinal,}\\ \bigcup_{\mu < \lambda} X(E|\mu) & \text{if } \lambda \text{ is a limit ordinal.}\end{cases}$$

Let $X(\boldsymbol{E})$ denote $\{X(E) \colon E \in \boldsymbol{E}\}$. $X(\boldsymbol{E})$ is α-directed by Lemma 4 below, so defining $X' = \varinjlim X(\boldsymbol{E})$ yields a subobject Y containing X. Let γ' be such that X' is γ'-generated. Note that by Lemma 2 above, γ' does not depend on X, or on X', but only on γ.

We show that X' is pure in Y. Given $G \in \boldsymbol{G}$ let $K' \rightrightarrows X' \otimes G \to Y \otimes G$ be a kernel pair diagram. $X' \otimes G = \varinjlim(X(\boldsymbol{E}) \otimes G)$ and since $X(\boldsymbol{E})$ is α-directed, $K' = \varinjlim K(\boldsymbol{E})$, where $K(E) \rightrightarrows X(E) \otimes G \to Y \otimes G$ is a kernel pair for each $E \in \boldsymbol{E}$. Let $X(EG) = (X(E))G$, obviously in $\boldsymbol{E}$. We have a diagram:

$$\begin{tikzcd}
K' \arrow[r, shift left] \arrow[r, shift right] & X' \otimes G \arrow[dr] & \\
K(EG) \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & X(EG) \otimes G \arrow[u] \arrow[r] & Y \otimes G\,. \\
K(E) \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & X(E) \otimes G \arrow[u] \arrow[ur] &
\end{tikzcd}$$

The diagram $K(E) \rightrightarrows X(E) \otimes G \to X(EG) \otimes G$ commutes by Lemma 2 above, so $K(E) \to K' \rightrightarrows X' \otimes G$ commutes. Since $K' = \varinjlim K(E)$, $K' \rightrightarrows X' \otimes G$ commutes, and $X' \otimes G \to Y \otimes G$ is a monomorphism for every $G \in \boldsymbol{G}$.

Given $Z \in \boldsymbol{X}$, $Z = \varinjlim_I G_i$ for some α-filtered family $\{G_i\}_{i \in I} \subseteq \boldsymbol{G}$. Thus $X' \otimes Z \to Y \otimes Z$ is a monomorphism, being $\varinjlim_I (X' \otimes G_i \to Y \otimes G_i)$.

4. Lemma. *α regular implies $X(E)$ is α-directed.*

Let $\boldsymbol{A}$ be a coalgebraic propable category over $\boldsymbol{X}$. Each object A in $\boldsymbol{A}$ has structure maps of the form $A \xrightarrow{\sigma A} A^{\sigma r}$. Let $[A]$ denote the set of these structure maps and let $\sigma A = \sigma$ whenever A is understood. If A and B are in $\boldsymbol{A}$, $A \otimes B$ has a natural $\boldsymbol{A}$-structure defined by:

$$A \otimes B \xrightarrow{\sigma A \otimes \sigma B} A^{\sigma r} \otimes B^{\sigma r} \cong (A \otimes B)^{\sigma r} .$$

Now assume that $\otimes$ preserves all colimits. Then the underlying functor $\boldsymbol{A} \to \boldsymbol{X}$ creates colimits and, hence, $\boldsymbol{A}$ is cocomplete. Let A be in $\boldsymbol{A}$ and let $X \to A$ be a pure $\boldsymbol{X}$-subobject of A. Then X is an $\boldsymbol{A}$-subobject if and only if $\sigma|X$ factors through $X^{\sigma r}$ for each $\sigma \in [A]$.

5. Proposition. *Let A be in $\boldsymbol{A}$ and let $X \xrightarrow{i} A$ be a γ-generated $\boldsymbol{X}$-subobject of A. Then there exists a γ-generated subobject of A, $\overline{X} \xrightarrow{j} A$ containing X such that $\mathrm{Im}(i\sigma) \subseteq \mathrm{Im}(j^{\sigma r})$ for all $\sigma \in [A]$, provided $\gamma > |[A]|$.*

Proof. Let $\boldsymbol{U}$ be as in the proof of Lemma 2. Then $\varinjlim \boldsymbol{U} = A$, so for any $n \in \mathbf{N}$,

$$A^n = \bigotimes_n \varinjlim \boldsymbol{U} = \varinjlim_{U_i \in \boldsymbol{U}} (U_1 \otimes \ldots \otimes U_n) = \varinjlim_{U \in \boldsymbol{U}} U^n$$

since the diagonal is cofinal. Because X is γ-generated and the colimit $\varinjlim_{U \in \boldsymbol{U}} U^{\sigma r} = A^{\sigma r}$ is γ-filtered, $\mathrm{Im}(i\sigma) \subseteq \mathrm{Im}(U_\sigma^\sigma \to A^{\sigma r})$ for some $U_\sigma \in \boldsymbol{U}$. Let $\overline{X} = \bigcup_{\sigma \in [A]} U_\sigma$.

Diagonalizing the processes of Proposition 5 and Theorem 3 we have:

6. Proposition. *There exists a cardinal $\gamma(A)$ such that each γ-generated $\boldsymbol{X}$-subobject of A (in $\boldsymbol{A}$) is contained in a $\gamma(A)$-generated pure $\boldsymbol{A}$-subobject of A.*

7. Corollary. *$\boldsymbol{A}$ has a set of generators.*

8. Corollary. *There exists a right adjoint to the forgetful functor $\boldsymbol{A} \to \boldsymbol{X}$.*

In fact $\boldsymbol{A}$ is cotripleable over $\boldsymbol{X}$, as can be easily shown by modifying the argument for vector spaces in [5] or for R-modules in [2]. It is also easy to see that the generators for A constructed above have rank in A, which is thus locally-presentable. In fact, $A(A, B)$ is a subset of $X(A, B)$ defined as the γ-limit of a certain set of maps of the form $X(A, B) \to X(A, B^n)$.

References

[1] M. Barr, Exact Categories, Springer Lecture Notes # 236 (1971) 1–120.
[2] M. Barr, Coalgebras over a Commutative Ring, J. Alg. 32 (1974) 600–610.
[3] P. Gabriel and F. Ulmer, Lokal präsentierbare Kategorien, Springer Lecture Notes # 221 (1971).
[4] S. MacLane, Categorical Algebra, Bull. A.M.S. 71 (1965) 40–106.
[5] D. Van Osdol, Coalgebras, Sheaves, and Cohomology, Proc. A.M.S. 33 (1972) 257–263.